The two isosceles triangles CJE and IJK, having equal angles at E and K respectively, are similar. Hence we have $JE : CE :: JK : IK$, or, using (1), (2) and (3)

$$h\sqrt{4-h^2} : (3h - h^3) :: h : (2h - h^3).$$

Hence

$$(2 - h^2)\sqrt{4-h^2} = 3 - h^2. \tag{4'}$$

Squaring, we get

$$(4 - 4h^2 + h^4)(4 - h^2) = 9 - 6h^2 + h^4,$$

and expanding,

$$16 - 20h^2 + 8h^4 - h^6 = 9 - 6h^2 + h^4.$$

Finally, transposing and simplifying, we obtain the author's *Heptagon Cubic:*

$$7 - 14h^2 + 7h^4 - h^6 = 0. \tag{5}$$

Solving by Horner's method, we find $h^2 = .7530203962821\ldots = \frac{3}{4}$, approximately.

Remark. It will thus be seen that while there is introduced a new line IJ, we dispense with the consideration of the line OC, and with both the consideration and the computation of the author's lines SE, SJ, SK and SC. As a result, the equation (4′) appears in a much simpler form than the author's equation (4).

The approximate construction of the heptagon may also be simplified as follows:

Let M, N and P be three consecutive vertices of an inscribed regular hexagon. Draw the chord MP and the radius ON, and let MP meet ON in R. Then MR is, approximately, the length h of the side of the regular inscribed heptagon. The reason is self-evident: approximately, $h = \frac{1}{2}\sqrt{3}$, and MP, as the side of a regular inscribed triangle, $= \sqrt{3}$, so that $MR = \frac{1}{2}\sqrt{3}$, and therefore $MR = h$, approximately.

A PROBLEM IN NUMBER THEORY.

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§ 1. When is the sum of the squares of two successive integers a perfect square? The following are examples:

$$3^2 + 4^2 = 5^2,\quad 20^2 + 21^2 = 29^2.\quad \text{The next is } 119^2 + 120^2 = 169^2.$$

The numbers 3, 20, 119, . . . are the terms of a series

$$0,\ 3,\ 20,\ 119,\ 696,\ \cdots\ u_n,\ u_{n+1}, \tag{1}$$

where

$$u_{n+1} = 6u_n - u_{n-1} + 2. \tag{2}$$

This may be proved as follows:

From (2), which is the relation between any three successive terms of (1), we may derive the relation between any two successive terms as follows:

From (2)

$$u_{n+1} + u_{n-1} = 6u_n + 2,$$

$$u_{n+1}^2 - u_{n-1}^2 = (6u_n + 2)(u_{n+1} - u_{n-1}),$$

$$(u_{n+1} - 1)^2 - 6u_n u_{n+1} = (u_{n-1} - 1)^2 - 6u_n u_{n-1},$$

Adding to each member $(u_n - 1)^2$, we have

$$(u_{n+1} - 1)^2 + (u_n - 1)^2 - 6u_n u_{n+1} = (u_n - 1)^2 + (u_{n-1} - 1)^2 - 6u_{n-1}u_n, \quad (3)$$

which is of the form $f(n) = f(n - 1)$.

Hence by induction, $f(n) = c$, a constant independent of n. By applying the first member of (3) to the terms 3 and 20, we find $c = 5$. Hence

$$(u_{n+1} - 1)^2 + (u_n - 1)^2 - 6u_n u_{n+1} = 5 \quad (4)$$

is the relation between any two successive terms of (1).

Solving (4) with respect to u_{n+1}, we have

$$u_{n+1} = 3u_n + 1 \pm 2\sqrt{2u_n^2 + 2u_n + 1}, \quad (5)$$

in which the lower sign is rejected since, otherwise, the right member would be less than u_n. It follows from (5) that

$$2u_n^2 + 2u_n + 1 = \text{a square},$$

that is,

$$u_n^2 + (u_n + 1)^2 = \text{a square},$$

one part of the result which was to be proved.

§ 2. From the terms of (1) we may write

$$\begin{aligned}
0^2 + 1^2 &= 1^2,\\
3^2 + 4^2 &= 5^2,\\
20^2 + 21^2 &= 29^2,\\
119^2 + 120^2 &= 169^2,\\
696^2 + 697^2 &= 985^2,\\
4059^2 + 4060^2 &= 5741^2,\\
23660^2 + 23661^2 &= 33461^2,\\
137903^2 + 137904^2 &= 195025^2,\\
803760^2 + 803761^2 &= 1136689^2,\\
\cdots \quad &\cdots\\
\cdots \quad &\cdots
\end{aligned}$$

The second members are terms of a series,

$$1,\ 5,\ 29,\ \cdots u_n,\ u_{n+1},$$

where

$$u_{n+1} = 6u_n - u_{n-1}.$$

§ 3. It remains to be shown that the terms of the series (1) are the *only* integers that satisfy the condition

$$N^2 + (N + 1)^2 = \text{a square}. \tag{6}$$

If from (4) we express u_n in terms of u_{n+1}, we have

$$u_n = 3u_{n+1} + 1 - 2\sqrt{2u_{n+1}^2 + 2u_{n+1} + 1}, \tag{7}$$

from which

$$u_{n-1} = 3u_n + 1 - 2\sqrt{2u_n^2 + 2u_n + 1}. \tag{8}$$

Consider the equation

$$y = 3x + 1 - 2\sqrt{2x^2 + 2x + 1}. \tag{9}$$

Then

$$\begin{aligned}2y^2 + 2y + 1 &= 2(3x + 1 - 2\sqrt{2x^2 + 2x + 1})^2 + 2(3x + 1 - 2\sqrt{2x^2 + 2x + 1}) + 1 \\ &= (4x + 2 - 3\sqrt{2x^2 + 2x + 1})^2.\end{aligned}$$

Hence if

$$2x^2 + 2x + 1 = \text{a square},$$

then also

$$2y^2 + 2y + 1 = \text{a square}.$$

That is, if x satisfies (6), so does y. Comparing (9) with (7) and (8), it appears that if $x = u_{n+1}$, $y = u_n$; and if $x = u_n$, $y = u_{n-1}$. And as y is an increasing function of x, since

$$\frac{dy}{dx} = 3 - \frac{4x + 2}{\sqrt{2x^2 + 2x + 1}} > 0,$$

it follows that if $u_n < x < u_{n+1}$, then $u_{n-1} < y < u_n$. That is, if there is an integer satisfying (6) between u_n and u_{n+1}, there is another such integer between u_{n-1} and u_n.

There is no integer satisfying (6) between the terms 3 and 20; hence there is none between 20 and 119, and consequently none between any two successive terms of the series (1).

§ 4. In the list of equations in § 2, it may be noticed that none of the successive integers end in 2, 5 or 8. As the final digits recur, it follows that

The sum of the squares of two successive integers, one of which ends in 2, 5 *or* 8, *cannot be a perfect square.*